

# Quasi-black holes: definition and general properties

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Objects that are on the verge of being extremal black holes but actually are distinct in many ways are called quasi-black holes. Quasi-black holes are defined here and treated in a unified way through the displaying of their properties. The main ones are (i) there are infinite redshift whole regions, (ii) the spacetimes exhibit degenerate, almost singular, features but their curvature invariants remain perfectly regular everywhere, (iii) in the limit under discussion, outer and inner regions become mutually impenetrable and disjoint, although, in contrast to the usual black holes, this separation is of a dynamical nature, rather than purely causal, (iv) for external far away observers the spacetime is virtually indistinguishable from that of extremal black holes. It is shown, in addition, that quasi-black holes must be extremal. Connections with black hole and wormhole physics are also drawn.

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## I. INTRODUCTION

A quasi-black hole (QBH) is neither a usual regular spacetime, as for instance a star, nor a black hole (BH). So what is it? Roughly speaking one can say that a QBH is an object on the verge of becoming an extremal BH but actually is distinct from it in many ways. In the present paper we show that, among other properties, perhaps the most striking ones that characterize a QBH are that there is an infinite redshift region, the spacetime exhibits degenerate features with the curvature invariants remaining perfectly regular everywhere, outer and inner regions become mutually impenetrable and disjoint, and for an external observer at infinity the spacetime is indistinguishable from that of an extremal black hole. Another interesting feature is that a QBH has to be extremal.

To try to understand how a QBH may arise, we note that, remarkably, contrary to the common case where instabilities set in much before a matter system reaches its own gravitational radius, there are some systems for which the gravitational radius can be approached in a sequence of static configurations. They were first noticed within the context of Majumdar-Papapetrou systems [1, 2] by Bonnor in [3–5] and are systems composed of extremal charged dust, where the energy density is equal to the charge density, with no pressure term, and joined to an asymptotically flat extremal Reissner-Nordström region. These systems are called Bonnor stars. In recent years, the interest in such objects was renewed due to further investigations on their properties, where it was found that they have very interesting properties such as the formation of a QBH state [6, 7]. The same set of properties had also been found in extended Bonnor stars [8], extremal systems with a more sophisticated density distribution. One can then say that QBHs can be thought of as the end state of a sequence, of gradually more compact, quasistatic appropriate Majumdar-Papapetrou configurations, such as Bonnor stars and extended Bonnor stars. The most surprising feature of all these systems is the fact that the limiting case, at the threshold of the formation of an event horizon, is very peculiar. Although to external observers the system looks like an extremal black hole, its internal properties, so to speak, are very different from what one could expect in the case of a usual BH. In the limit, instead of an extremal BH one has a QBH, and instead of an event horizon one has a quasihorizon. This then expands the existing taxonomy

of relativistic objects, adding to it something that is neither a usual regular spacetime, a star, nor a BH, it is a QBH. There are other systems that display QBHs. In self-gravitating Higgs magnetic monopole systems, a seemingly different system, it was also found, in a totally independent way, that in a certain well defined limit a QBH appears as a natural state, and it was indeed within these studies that the term QBH was coined [9, 10]. The similarity of the properties of the Bonnor stars and gravitational magnetic monopoles was clearly recognized in [7]. Both kinds of systems look quite physical. For example, the Bonnor star system can be realized when a sphere of neutral hydrogen has lost a fraction  $10^{-18}$  of its electrons, while magnetic monopoles should be formed if standard grand unified theories prove to be correct. In addition, and surprisingly, similar objects with QBH properties, were found for composite spacetimes even in the case of pure electrovacuum [11] (see also [12]). These vacuum systems are composed of an exterior Reissner-Nordström part glued to an inner Bertotti-Robinson spacetime (see [13–18]), or of an exterior Reissner-Nordström part glued to an inner Minkowski spacetime [19]. It is interesting to note that in a certain sense some of these systems realize the idea of charge without charge [20].

There are at least eight related subjects connected to QBHs. Not all of them will be analyzed in detail, since that would take us far a field. The first connection is with naked BHs [21–24] (see also [25] for the issue of the behavior of the quasilocal energy and momentum under boosts from a static frame to a free-falling one). Naked BHs have diverging Riemann tensors in certain physically meaningful frames, which in turn relates them to the singular or regular character of a spacetime and the cosmic censorship hypothesis [26]. As we will see, QBHs have this naked property. The second connection is with the end state of an extremal matter configuration. One can ask whether a QBH can be attained physically. Can a QBH configuration be reached through a finite number of steps from a regular configuration? This is related to whether an analogue of the third law of BH thermodynamics (see, e.g., [27]) is enforced or not for QBHs. The third connection is with the instabilities that might set in before the gravitational radius is reached. This is related to the Buchdahl limit [28], i.e., the minimum radius to mass ratio  $r_0/m$  that a stable configuration can have, where  $r_0$  and  $m$  are the radius and mass of the configuration, respectively. For perfect fluid spheres it is  $r_0/m \geq 9/4$ , while for charged spheres the ratio decreases, it goes to  $r_0/m \geq 1$  precisely in

the case of extremal charged dust [29, 30]. The fourth connection is with the hoop conjecture [31], since it seems that QBHs grossly violate it. The conjecture states that a BH forms when matter of mass  $M$  is compacted within a given definite hoop, in [31] taken to be  $\sim 4\pi M$  ( $G = C = 1$ ), and shown later in [32] that the hoop should be reduced for extremal charged matter to  $\sim 2\pi M$ . But as it will be shown, for extremal matter a BH never arises, instead a QBH forms in the limit. The fifth connection is with the no hair theorems. It was conjectured by Wheeler that BHs should have no hair, in particular no electromagnetic hair, a conjecture that has been verified [33, 34]. On the other hand QBHs have the feature that they may have some electromagnetic hair [8], adding to the list of distinct properties between both objects. The sixth connection is with Bardeen BHs [35, 36], i.e., BHs that have a kind of magnetic charged matter inside the horizon, and have no singularities inside. Following a theorem by Borde [37] this means the topology inside the horizon is different from the usual one. Now, the configurations we are studying are neither usual BHs nor Bardeen BHs, they are QBHs. They have quasihorizons and the Kretschmann scalar is finite inside, although, as we will see, this does not exclude other degenerate features. So, it appears that in order to avoid a true horizon without a singularity inside with a consequent change of topology, the object opts to form a QBH, instead of a Bardeen-type BH. The seventh connection is with objects that mimic BHs. For instance, wormholes (see, e.g., [38]) can be good mimickers of BH properties [39]. Although, QBHs and BHs share many properties from the viewpoint of an external observer, the full study of this subject has not been done. The eighth connection is with the entropy issue. For the usual BHs one does not yet know for sure where are the degrees of freedom and thus how their entropy arises (see, e.g., [40]). For QBHs it seems that the entropy comes from the entangled fields hidden beyond the quasi horizon [41]. There are possible connections with other subjects, like gravitational collapse (which has not been studied for the case of extremal matter) and vacuum polarization effects, to name two.

In this work we obtain and analyze the geometric and physical properties of a QBH. The paper is organized as follows. In section II a definition of a QBH is given. In section III the properties of QBHs are displayed in several instances. Initially, we study Bonnor stars, both truncated and extended, then we analyze gravitational magnetic monopoles, and finally we study glued extremal vacua. All these instances of QBHs show a number of similar

properties. In section IV we prove an important theorem that states that QBHs have to be extremal. In section V we discuss the relationship between regular and singular features in QBHs spacetime and whether the QBH state can be physically attained. Finally in section VI we draw some interesting conclusions.

## II. DEFINITION OF QUASI BLACK HOLES

The fact that so different kinds of physical systems like extremal dust, Yang-Mills–Higgs matter, and composite vacuum systems, may exhibit the same qualitative features suggests that the unusual properties of QBHs can be explained in an unified manner. So first we define what a QBH is, and then we investigate in detail the properties of such a system in the various instances.

A QBH can be defined as an object with the following properties. Consider the static spherically symmetric metric, often written as

$$ds^2 = -B(r) dt^2 + A(r) dr^2 + r^2 d\Omega^2, \quad (1)$$

where  $r$  is the Schwarzschild radial coordinate,  $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ , and  $B(r)$  and  $A(r)$  are metric potentials. It is useful to define a new metric potential  $V$  through

$$V(r) = \frac{1}{A(r)}. \quad (2)$$

Let an inner matter configuration, with an asymptotic flat exterior region, exist with the properties (a) the function  $V(r)$  attains a minimum at some  $r^* \neq 0$ , such that  $V(r^*) = \varepsilon$ , with  $\varepsilon \ll 1$ , this minimum being achieved either from both sides of  $r^*$  or from  $r > r^*$  alone, (b) for such a small but nonzero  $\varepsilon$  the configuration is regular everywhere with a nonvanishing metric function  $B$ , at most the metric contains only delta-function like shells, and (c) in the limit  $\varepsilon \rightarrow 0$  the metric coefficient  $B \rightarrow 0$  for all  $r \leq r^*$ . These three features define a QBH. Note that although the above definition of QBHs relies on the coordinate system and metric coefficient  $V$  given in equations (1)-(2), actually, this definition can be done in a form invariant under the choice of the radial coordinate. Indeed, it is sufficient to replace  $V$  by  $(\nabla r)^2$ . In the Schwarzschild coordinates of equation (1) one has  $(\nabla r)^2 = V$ .

In turn, these three features entail some nontrivial consequences: (i) there are infinite redshift whole regions, (ii) when  $\varepsilon \rightarrow 0$ , a free-falling observer finds in his own frame infinitely large tidal forces in the whole inner region, showing some form of degeneracy, although the spacetime curvature invariants remain perfectly regular everywhere, (iii) in the limit, outer and inner regions become mutually impenetrable and disjoint, and one can also show that (iv) for external far away observers the spacetime is virtually indistinguishable from that of extremal black holes. In addition, QBHs must be extremal. The QBH is on the verge of forming an event horizon, but it never forms one, instead, a quasihorizon appears. For a QBH the metric is well defined and everywhere regular. However, properties, such as when  $\varepsilon = 0$ , QBH spacetimes become degenerate, almost singular, have to be examined with care.

### III. PROPERTIES OF QUASI-BLACK HOLES

Now, we study the three different examples of QBH behavior separately (namely, extremal charged dust, Yang-Mills–Higgs matter, and composite vacuum systems), to show how the same features reveal themselves in these different circumstances.

#### A. Extremal charged dust and Bonnor stars

Within extremal charged dust there are two different cases worth of study, namely the ones studied by Bonnor [3–7] and the ones studied by Lemos and Weinberg [8], both systems belong to the Majumdar-Papapetrou class [1, 2].

1. *Bonnor stars: bounded distribution of extremal dust matched to an electrovacuum at  $r = r_0$  (with  $r_0 > m$ ) [3–7]*

*Generic properties:*

The radius  $r_0$  is the boundary of the star. Inside there is matter, outside there is vacuum. So, in the region  $r > r_0$  the metric potential  $V$  has the usual form for the extremal Reissner-Nordström BH,

$$V^{\text{RN}} = \left(1 - \frac{m}{r}\right)^2, \quad (3)$$

where  $m$  is the total mass. For  $r \leq r_0$  it is described by a Majumdar-Papapetrou type solution which in Schwarzschild-like coordinates can be written as

$$V = \left(1 - \frac{\mu(r)}{r}\right)^2, \quad (4)$$

with, the mass density  $\rho$  and the function  $\mu$  being connected through

$$4\pi\rho = \frac{\mu'}{r^2} \left(1 - \frac{\mu}{r}\right). \quad (5)$$

The function  $\mu(r)$  can be interpreted as the proper mass enclosed within a sphere of a radius  $r$ . Similarly, we define  $e(r)$  as the proper charge enclosed within a sphere of a radius  $r$ . For Majumdar-Papapetrou systems  $\mu(r) = e(r)$ . We want to glue smoothly both regions, so  $\mu(r_0) = m$  and for  $r \leq r_0$ ,

$$\sqrt{B(r)} = \sqrt{B^{\text{RN}}(r_0)} \exp(\nu), \quad (6)$$

where,

$$\nu = \int_{r_0}^r dr \frac{\mu}{r^2 \left(1 - \frac{\mu}{r}\right)}. \quad (7)$$

This guarantees that on the boundary,  $\sqrt{B(r_0 - 0)} = \sqrt{B(r_0 + 0)}$ .

*Proper spatial distance:*

The proper distance can be written as

$$l = \int dr \frac{1}{\sqrt{V(r)}} = \int dr \frac{1}{\left(1 - \frac{\mu}{r}\right)} = \int dr \frac{\mu'}{4\pi\rho r^2}. \quad (8)$$

If  $\rho$  remains finite and nonzero in the quasihorizon limit  $r_0 \rightarrow m$ , like in the special examples of [3–7], one can obtain from (5) that  $\mu \approx m - [8\pi\rho(m)m^3]^{1/2}(r - m)^{-1/2}$  near  $r = m$  and, thus, the integral (8) converges and the proper distance from any interior point to the boundary, from the inside, remains finite accordingly. In terms of the potential  $V(r)$  we can see this by noting that when  $V(r)$  has a single root the integral in (8) is finite, when it has a double root the integral behaves logarithmically and yields an infinite result. So, from the inside one has  $\lim_{r_0 \rightarrow m}^{r \rightarrow r_0} V'(r) = -8\pi\rho(r_0)r_0|_{r_0=m} < 0$ , the root is simple and the proper distance is finite. On the other hand, from the outside,  $V'(r)|_{r_0 \rightarrow m} \rightarrow 0$ , one has thus a double root in the limit, and the proper distance is infinite, yielding a semi-infinite

throat from the outside, which is a well known result for the extremal Reissner-Nordström geometry.

*Motion of massive and massless particles:*

From the inside to the outside, the existence of an impenetrable barrier: Now, let us consider the motion of particles in this spacetime. In doing so, an important question concerns the transition from the inner region to the outer one. In the interior the suitable time variable measured by a static observer can be obtained by rescaling the time  $t$ , such that  $t = \frac{\tilde{t}}{\sqrt{B^{\text{RN}}(r_0)}}$ . So  $\sqrt{\tilde{B}} \equiv \frac{\sqrt{B}}{\sqrt{B^{\text{RN}}(r_0)}}$  is finite. Now, if a timelike particle is emitted in the radial upward direction with the finite energy  $\tilde{E}$  one can easily find from the conservation law that the proper time  $\tilde{\tau}$  is equal to  $\tilde{\tau} = \int d\tilde{l} \frac{\sqrt{\tilde{B}}}{\sqrt{\tilde{E}^2 - \tilde{B}}}$ , where, with respect to time  $\tilde{t}$ , we have put the proper mass of a timelike particle equal to one. Thus, the particle reaches the border in a finite proper time  $\tilde{\tau}$ . The quantity  $\sqrt{\tilde{B}}$  is finite everywhere inside and is equal to unity on the border. However, outside one has that  $\sqrt{\tilde{B}} = \frac{(r-m)r_0}{(r_0-m)r}$  grows without bound when one takes the limit  $r_0 \rightarrow m$  for any  $r > m$ . When denominator in the equation above vanishes, it gives a turning point at  $r_1 = \frac{mr_0}{r_0 - \tilde{E}(r_0 - m)}$ . In the limit  $r_0 \rightarrow m$  we have also  $r_1 \rightarrow m$  for any finite  $\tilde{E}$ . Thus, the boundary between the matter and vacuum regions, acts like an infinite barrier which prevents particles from penetrating into the outer region from inside. For zero rest mass particles, like photons, moving radially, the affine parameter  $\lambda$  is given by  $\lambda = \tilde{\omega}^{-1} \int d\tilde{l} \sqrt{\tilde{B}}$ , where  $\tilde{\omega}$  is the photon frequency measured with respect to the time  $\tilde{t}$ . Then in the outer region one has  $\lambda(r) - \lambda(r_0) = \frac{(r-m)r_0}{(r_0-m)r}$ . This difference becomes infinite in the limit  $r_0 \rightarrow m$  for any  $r > m$ . As a result, again the boundary in the limit under discussion acts as an impenetrable barrier. Thus it also acts as a lightlike infinity.

From the outside to the inside, shrinking interval of proper time, tidal forces and naked behavior: (i) Shrinking interval of proper time - it follows from the above formulas that in the limit  $B^{\text{RN}}(r_0) \rightarrow 0$  the finite interval in time  $\tilde{t}$  correspond to infinitely delayed intervals in time  $t$ . However, if one calculated the proper time for an infalling particle moving with the energy  $E$  from the outside (which is defined with respect to the time at infinity, without rescaling) it follows from  $\tau = \int dl \frac{\sqrt{B}}{\sqrt{E^2 - B}}$  that  $\tau \rightarrow 0$  between any two points inside since  $B \rightarrow 0$  there, while the proper distance is finite for bounded Bonnor stars as is explained after



eq. (8). Manifestations of these general properties for self-gravitating monopole spacetimes were discussed in [10]. (ii) Tidal forces and naked behavior - to understand the existence of naked behavior for these systems we have to compute the Riemann tensor in a freely falling frame. First we compute it in the static coordinate frame. Consider then the behavior of the nonvanishing Riemann tensor components. One has,  $R_{\hat{0}\hat{r}}^{\hat{0}\hat{r}} \equiv K = -V \frac{\sqrt{B}''}{\sqrt{B}} - \frac{V'}{2} \frac{\sqrt{B}'}{\sqrt{B}}$ ,  $R_{\hat{0}\hat{\theta}}^{\hat{0}\hat{\theta}} = \bar{K}(r) = -\frac{V}{r} \frac{\sqrt{B}'}{\sqrt{B}}$ ,  $R_{\hat{\phi}\hat{\theta}}^{\hat{\phi}\hat{\theta}} \equiv F(r) = \frac{1}{r^2}(1 - V)$ ,  $R_{\hat{\theta}\hat{r}}^{\hat{\theta}\hat{r}} \equiv \bar{F}(r) = -\frac{V'}{2r}$ . One can then obtain directly that all these components remain finite in the inner region in the limit  $\sqrt{B(r_0)} \rightarrow 0$ . Indeed, it follows from (4)-(6) that the quantities defined above are given by,  $K = \frac{2\mu}{r^3} - 3\frac{\mu^2}{r^4}$ ,  $\bar{K}(r) = -\frac{\mu}{r^3}$ ,  $F = \frac{1}{r^2} \left( 2\frac{\mu}{f} - \frac{\mu^2}{r^2} \right)$ ,  $\bar{F}(r) = -8\pi r \rho$ . So for finite  $\rho$  and  $\mu$ , the above quantities are obviously finite everywhere in the inner region including the boundary and origin. Correspondingly, the Kretschmann scalar is finite and the geometry is regular in spite of the fact that the metric function  $\sqrt{B}$ , suited to the time variable of an asymptotically flat observer, vanishes everywhere in the inner region. Having computed the Riemann tensor in the static coordinate frame we can now go on to a free-falling frame. Here, the situation becomes more subtle. We have now enhancement of the curvature components. To see this, write first,  $Z \equiv (\bar{F} - \bar{K})$ . Then,  $\tilde{Z} = Z(2\frac{E^2}{B} - 1)$ , where  $E$  is the energy of the freely falling particle, representing the freely falling particle frame. So, one sees that in the limit  $\sqrt{B} \rightarrow 0$ , these components of the curvature tensor and the corresponding tidal forces grow without bound. Thus, we encounter behavior typical of naked BHs [21, 22] (see also [23–25]), although in the present case we have QBHs instead of BHs. Note, in passing, that naked behavior is consistent with the regularity of the geometry in the static frame since in the free-falling frame different terms enter the expression in the Kretschmann scalar with different signs and may mutually cancel.

#### *Redshift:*

Bonnor stars, in the limit of QBH formation, display infinite redshift phenomenon as shown in [3–7], where it is assumed that the frequency is measured with respect to time  $t$  at infinity. However, we have seen that two scales of time,  $t$  and  $\tilde{t}$ , are relevant for the systems under discussion.

In terms of the time  $t$ , the outer time, the product  $\omega \sqrt{B(r)} = \omega_c$  remains constant on

each ray during the propagation of light in a static gravitational field, where here  $\omega_c$  is some constant frequency. Note that  $\omega_c$  is a frequency measured with respect to time  $t$ ,  $\omega$  is the frequency measured with respect to the proper time at a given point  $r$ . Since at infinity  $B = 1$ , one obtains that  $\omega_c$  is  $\omega(r \rightarrow \infty) \equiv \omega_\infty$ , so that one can write  $\omega\sqrt{B(r)} = \omega_\infty$ . A distant observer would register an infinite redshift ( $\omega_c \rightarrow 0$ ) if an emitted particle had a finite  $\omega$  inside the matter since  $B \rightarrow 0$  there in the QBH limit. Only high-frequency photons with infinite  $\omega$  inside the quasihorizon but finite  $\omega_\infty$  can escape to infinity. This occurs for any Bonnor star whose boundary gets arbitrarily close to the horizon ( $B^{\text{RN}}(r_0) \rightarrow 0$ ), this property being model-independent.

In terms of the time  $\tilde{t}$ , the inner time, an observer uses rather the equality  $\omega\sqrt{\tilde{B}} = \tilde{\omega}_c$ , where  $\sqrt{\tilde{B}} = \frac{B}{\sqrt{B^{\text{RN}}(r_0)}}$ , as above, and  $\tilde{\omega}_c = \frac{\omega_\infty}{\sqrt{B^{\text{RN}}(r_0)}}$ . The observer does not encounter an infinitely large redshift since in the inner region  $\tilde{\omega}_c$  and  $\tilde{B}$  remain finite and nonzero, even in the QBH limit when  $B^{\text{RN}}(r_0) \rightarrow 0$ . However, we have seen before that the latter property causes an infinite barrier for particles moving outward.

Thus, both properties (infinite redshift for an inner signal, emitted inside and registered by an observer at infinity, and impenetrable barrier for particles moving from the inner to the outer region) are different consequences of the same property  $B^{\text{RN}}(r_0) \rightarrow 0$ .

#### *Other considerations: the end state*

The QBH can be considered as the end state of a sequence of ever more compact Bonnor stars. There is no way in which one can get a more compact object from it, or somehow turn it into an extremal BH. Whether this end state can be achieved by a physical process is a thorny issue that will be discussed towards the end of this article.

#### *Example:*

We demonstrate now, using an explicit example of a Bonnor star given in [5], what happens to the metric in the quasihorizon limit. The metric of any spherically-symmetrical Majumdar-Papapetrou system can be written in isotropic coordinates as (see [1, 2]),

$$ds^2 = -B dt^2 + B^{-1} (dR^2 + R^2 d\Omega^2) \quad (9)$$

where the radial coordinate  $R$  is related to the Schwarzschild coordinate  $r$  of equation (1)

by

$$R = r \sqrt{B}. \quad (10)$$

From [5], defining a new potential  $U(R)$  as  $U = 1/\sqrt{B}$ , a good choice for the internal and external  $U$ ,  $U^I$  and  $U^E$  respectively, is

$$U^I = 1 + \frac{m}{R_0} + \frac{m(R_0^n - R^n)}{nR_0^{n+1}}, \quad 0 \leq R \leq R_0, \quad (11)$$

$$U^E = 1 + \frac{m}{R}, \quad R \geq R_0 > 0, \quad (12)$$

where  $m$  is the mass of the configuration,  $R_0$  is the boundary of the star, and  $n$  is a free exponent, with  $n \geq 2$ ,  $n = 2$  being a typical case. The extremal charged dust occupies the region  $0 \leq R \leq R_0$ . For  $R > R_0$  the metric represents an external extremal Reissner-Nordström metric. In this outer region the relation between  $r$  and  $R$  is simple,  $r = R + m$ . Then the boundary areal radius  $r_0$  is given by  $r_0 = m + R_0$ . When  $R_0 \rightarrow 0$  the areal radius  $r_0$  of the boundary approaches that of the quasihorizon as closely as one likes, with the dust density remaining finite everywhere inside, including the boundary. Let us take the next step to obtain the limiting metric explicitly. It is convenient to make the following substitutions for the interior metric,

$$R = R_0 x, \quad 0 \leq x \leq 1, \quad (13)$$

$$t = \frac{mT}{R_0}, \quad (14)$$

where  $x$  and  $T$  are new coordinates. Then, the limit  $R_0 \rightarrow 0$  can be taken safely and we obtain the metric of the interior,

$$ds^2 = - \left(1 + \frac{1}{n} - \frac{x^n}{n}\right)^{-1} dT^2 + m^2 \left(1 + \frac{1}{n} - \frac{x^n}{n}\right)^2 (dx^2 + x^2 d\Omega^2). \quad (15)$$

It is regular everywhere inside but incomplete for  $x \leq 1$ . It can be extended at least up to a singular  $x_s$ , given by  $x_s = (n+1)^{1/n} > 1$ , but this singularity has nothing to do with our original system. Now, it is seen from (14) that, indeed, an infinite redshift occurs in the limit  $R_0 \rightarrow 0$  since finite intervals of  $T$  correspond to infinitely growing intervals of  $t$ . This mismatch in time scales gives a clear example of why particles from the inside cannot penetrate to the outside.

We can observe one more important feature here. It is essential that at  $x = 1$  (defining the boundary between dust and vacuum) the metric (15) has no horizon. Meanwhile, the outer metric represents an extremal Reissner-Nordström BH with the metric

$$ds^2 = - \left(1 - \frac{m}{r}\right)^2 dt^2 + \left(1 - \frac{m}{r}\right)^{-2} dr^2 + r^2 d\Omega^2, \quad (16)$$

and has a horizon at the boundary in this limit. Therefore, we cannot match smoothly the two geometries: the surface  $r = m$  (also given by  $x = 1$ ) is timelike when seen from inside and is lightlike when seen from outside. One may try to reconcile these two features by adopting the original time coordinate  $t$  inside as well. Then, in the limit  $R_0 \rightarrow 0$  the interval along  $r = m$  does indeed become null from inside. However, this is achieved at the expense of the metric becoming degenerate inside, since the term in  $dt^2$  vanishes everywhere in the inner region. Thus, in any case, spacetime as whole exhibits singular, degenerate, features.

2. *Bonnor stars extended: continuous distribution of extremal charged dust that asymptotes to the extremal Reissner-Nordström geometry [8]*

*Generic properties:*

In [8] Bonnor stars were modified, so that instead of having a boundary where the charged extremal dust and the extremal Reissner-Nordström vacuum match, one has now a continuous, extended, distribution of extremal charged dust which asymptotes to the extremal Reissner-Nordström geometry. This type of distributions is specially interesting since it allows for cases where there is a kind of hair when the QBH is forming, although these cases are not going to be discussed here. Now, it is useful to rewrite the metric (1) given in the Schwarzschild coordinate  $r$ , into an isotropic form (9), given in the coordinate  $R$ .

In [8], in these coordinates, the trial distribution is given by the following form of the potential

$$\sqrt{B} = \frac{z}{z + q}, \quad (17)$$

where

$$z \equiv \sqrt{R^2 + c^2}, \quad (18)$$

$c$  is a constant that can be chosen arbitrarily, and  $q$  can be thought of as the total charge, as we will see below. One can also find that the potential  $V$  defined in (2) is given by

$$\sqrt{V} = \frac{z^3 + qc^2}{z^2(q+z)}. \quad (19)$$

Then, the metric can be rewritten as

$$ds^2 = - \left( \frac{z}{z+q} \right)^2 dt^2 + \frac{(q+z)^2}{z^2 - c^2} dz^2 + \frac{(z^2 - c^2)(z+q)^2}{z^2} d\Omega^2, \quad (20)$$

valid for  $z \geq c$ . The density is then given by

$$\rho = \frac{3qc^2}{4\pi z^2 (q+z)^3}. \quad (21)$$

Then, one obtains that a quasihorizon forms at  $r^*$ , such that for  $c \ll q$  one has  $R = R^* \simeq q \left( \frac{2c^2}{q^2} \right)^{1/3}$ . The explicit asymptotic behavior near the quasihorizon reads,

$$\sqrt{B} = 2^{1/3} \left( \frac{c}{q} \right)^{2/3} + \frac{2}{3} \frac{(r-r_*)}{q} + \frac{2^{2/3}}{9c^{2/3}q^{4/3}} (r-r_*)^2 \dots, \quad (22)$$

$$V = \varepsilon + \frac{2(r-r_*)^2}{q^2} + \dots, \quad (23)$$

where,

$$\varepsilon = \frac{9}{2^{4/3}} \left( \frac{c}{q} \right)^{4/3}, \quad (24)$$

and in this limit,  $r^* = q$ . So, near the formation of the QBH, for  $c \rightarrow 0$ , one finds there are three characteristic regions. They are:

(I) The inner core region  $r \lesssim c$ : Here it is convenient to make the substitution  $z = cy$  and take the limit  $c \rightarrow 0$  afterward. Then, rescaling time as

$$t = \frac{q}{c} \tilde{t}, \quad (25)$$

and making one more substitution

$$\cosh u \equiv y = \frac{z}{c}, \quad (26)$$

one obtains

$$ds^2 = q^2 \left( -\cosh^2 u d\tilde{t}^2 + du^2 + \tanh^2 u d\Omega^2 \right). \quad (27)$$

Thus the metric is everywhere regular. If one allows  $u \rightarrow \infty$ , it becomes geodesically complete and asymptotically approaches the Bertotti-Robinson metric [13, 14].

(II) The vicinity of the quasihorizon  $r = r^* = q$ : Then, it is convenient to make the rescaling to the coordinates  $T$  and  $\eta$ ,

$$T = z^* \frac{t}{q}, \quad (28)$$

and

$$\eta = \frac{z}{z^*}, \quad (29)$$

where  $z^* = z(r^*)$  and  $\eta \leq 1$  corresponds to the inner region. Then one can find by direct substitution that in the limit  $c \rightarrow 0$  the metric takes the form  $ds^2 = -\eta^2 dT^2 + q^2 \left( \frac{d\eta^2}{\eta^2} + d\Omega^2 \right)$ , and defining a new radial coordinate  $l$  by  $\eta = \exp\left(\frac{l}{q}\right)$  with  $l < 0$ , one has

$$ds^2 = -\exp\left(\frac{2l}{q}\right) dT^2 + dl^2 + q^2 d\Omega^2. \quad (30)$$

This metric is nothing else than the extremal version of the Bertotti-Robinson metric [13, 14]. The region with  $r \neq r^*$  is simply removed from the manifold. The coordinate  $l$  can now be extended into its full range, i.e.,  $-\infty < l < \infty$ . As is known, the Bertotti-Robinson spacetime is geodesically complete and, through yet another coordinate transformation, can be cast into a form where the horizon is absent (see, e.g., [15]). It is instructive to note that the Bertotti-Robinson metric can be obtained also as an extremal limit of a nonextremal Reissner-Nordström spacetime. However, in that case the resulting metric takes the form of the nonextremal version of Bertotti-Robinson metric [17, 18]. Note that, actually, regions I and II represent two different subregions of the inner region inside the quasihorizon. If one makes the substitution  $y = \eta \frac{z^*}{c}$ , it becomes clear that (27) transforms to (30), provided  $\eta \gg \frac{c}{z^*} \sim c^{1/3}$ .

(III) The region  $r > r^*$ : Here one can take the limit in (20) directly and obtain the extremal Reissner-Nordström metric. As is known, the region  $r > r^*$  represents only part of the extremal Reissner-Nordström geometry (16).

We see, that from a formal viewpoint, the three different spacetimes that arise from a single one, when the QBH forms, illustrate the fact that the result of taking the appropriate

limit depends strongly on how the coordinates are involved in it [16]. Indeed, for very small but nonzero  $c$  we have three distinct regions in the whole spacetime, a spacetime that possesses no horizon. Each of those regions approaches the corresponding form in its domain of validity: region (I) represents the inner core region, region (II) gives the vicinity of the quasihorizon, and region (III) corresponds to the outer solution. The spatial geometry for  $r$  close to  $r^*$  represents an extended throat on both sides of the quasihorizon. The energy density  $\rho(r^*) \sim c^{2/3} \rightarrow 0$  [8]. In the limit  $c = 0$ , each of the three regions looks incomplete in the original range of coordinates but can be made complete after extension and proper continuation of coordinates into the whole region. Similarly to the example (9)-(12), one can observe from (30) that the surface  $r = r^*$  looks timelike from inside but lightlike outside, so a smooth matching is impossible.

We also note that the Bertotti-Robinson spacetime can appear as a result of two different limiting procedures, by taking a special portion of the extremal Reissner-Nordström metric and taking an appropriate limit, or by taking a special limit of the QBH case under discussion. For further details see Appendix A.

#### *The total mass:*

It is instructive to compare the contribution of the inner and outer regions to the total mass. Using the usual formulas for the energy density and relationship between  $r$  and  $R$  one can obtain that the proper mass  $m_p = 3qI$ , where  $I$  is given by,  $I = \int_0^\infty \frac{dy y^2}{(1+y^2)^{5/2}} = \frac{1}{3}$ . So,  $m_p = q$ . Generically, for Majumdar-Papapetrou systems the proper mass is equal to the electric charge. So  $q$  has the meaning of total charge and does not depend on  $c$ . From the calculation, one also finds that the major contribution comes from the inner region,  $0 \leq y \leq y^*$ , where  $y^* \equiv \frac{R^*}{c} = 2^{1/3} \left(\frac{q}{c}\right)^{1/3} \gg 1$ . The contribution from the outer region is of the order  $\frac{1}{2y^*}$  and becomes negligible in the limit  $c \rightarrow 0$ . The same is true for the ADM mass  $m$ . Thus, the competition from the two factors, infinite proper volume (due to the extended throat) and vanishing energy density, results in finite proper and ADM masses,  $m_p$  and  $m$ , respectively. The mass is concentrated under the quasihorizon. Indeed, it is seen from (21) that in the limit  $c \rightarrow 0$  the density  $\rho \rightarrow 0$  everywhere except in the region of small  $z \sim c$  near the origin.

*The curvature tensor and impenetrability:*

From region I to II and vice versa: As the inner region I is geodesically complete in the QBH limit and is at infinite proper distance from the quasihorizon, there is really no question about penetrability from I to II (which is adjacent to the quasihorizon) or III and vice versa. Indeed, geodesics from region I can never reach regions II and III.

From the inner region II to III (i.e., from the vicinity of the quasihorizon to the outside) and vice versa: Near the quasihorizon  $z = z^*$ , the components of the curvature tensor, following the previous adopted nomenclature, are  $K(r^*) = -\frac{1}{q^2}$ ,  $\bar{K}(r^*) = O(c^{2/3}) \rightarrow 0$ ,  $F(r^*) = \frac{1}{q^2}$ ,  $\bar{F}(r^*) = 0$ . Thus, in this sense the geometry is perfectly regular. However, in the free-falling frame, the quantity  $\tilde{Z}$  is of the order  $c^{-2/3}$  and diverges. So, a particle cannot penetrate from the outside to inside because infinite tidal forces appear, exactly in the manner it was explained above while discussing the pure Bonnor stars. In addition, the arguments presented previously for the pure Bonnor stars, show that a particle with a finite energy measured with respect to rescaled time  $T$  of region II cannot penetrate from the inner region to the outer one. As a result, regions II and III are mutually impenetrable.

*Other considerations:*

Generalizing the approach of [8], we can notice that for a continuous distribution of matter QBHs should always exist provided

$$\rho(r_*) \sim p_r(r_*) \leq O(\sqrt{\varepsilon}), \quad (31)$$

where  $p_r$  is the radial pressure,  $\sqrt{B(r^*)} \rightarrow 0$  and  $V = \varepsilon + a(r - r^*)^2 + \dots$ , with  $a$  a constant and  $\varepsilon \rightarrow 0$ . These properties indeed hold for the extremal dust solutions considered above [8]. It follows from Einstein equations that the metric function  $\sqrt{B}$  obeys,  $\sqrt{B} = \sqrt{V} \exp(\psi)$ , where  $\psi = 4\pi \int dr \frac{r(\rho + p_r)}{V}$ . Then an elementary evaluation shows that, on the quasihorizon,  $\psi$  remains finite in this limit due to the property (31). This entails that  $\sqrt{B(r^*)} \sim \sqrt{\varepsilon} \rightarrow 0$ . Since for QBHs, and in particular for Majumdar-Papapetrou dust, one has  $\frac{d\sqrt{B}}{dr} > 0$  (see Appendix B), this also means that  $\sqrt{B} \rightarrow 0$  for all  $r < r^*$ , and we return to the situation discussed above. However, without knowing the details of the system, one cannot state in advance whether or not the entire inner region will be regular.



## B. Yang-Mills–Higgs matter and gravitational magnetic monopoles [9, 10]

The 't Hooft-Polyakov magnetic monopole, with a global magnetic charge, is a solution of the Yang-Mills–Higgs system with no gravity. When one couples gravitation, new important features arise. This Einstein–Yang-Mills–Higgs system possesses regular self-gravitating solutions for a range of parameters. In addition, for a sufficiently massive monopole the system turns into an extremal configuration. It was noted in [9, 10] that such an extremal configuration is a QBH. Indeed, in those works it was coined for the first time the word QBH, to distinguish such an object clearly from an extremal BH. Such a magnetic QBH develops then an extremal quasihorizon, with all the nontrivial matter fields inside it. For our purposes here we note that the metric used for the gravitational magnetic monopoles is of the type given in equation (1), and that the numerical calculations carried out in [9, 10] show that  $\sqrt{B} \sim \varepsilon^q$ , where  $q$  ranges between 0.7 and unity, and that  $\tilde{Z} \sim \varepsilon^{-2q}$ , where  $\tilde{Z}$ , defined in the previous section, is a quantity related to the tidal forces in a free-falling frame. In turn, this implies that in a static frame the quantity  $Z$  is regular, but in a free-falling frame  $\tilde{Z}$  diverges. Thus, we have again the combination of a perfectly regular geometry with a naked-type behavior inside the entire inner region, as was observed in [9, 10]. The other properties of QBHs discussed in the previous subsection follow through a comparison between the properties of the Yang-Mills–Higgs system with its gravitational magnetic monopoles and the corresponding QBHs and the much simpler Majumdar-Papapetrou system with its Bonnor stars, along the lines of [7].

## C. Vacuum with a surface layer: gluing between the extremal Reissner-Nordström and other metrics

### 1. Gluing between the extremal Reissner-Nordström and Bertotti-Robinson metrics [11, 12]

#### *Generic properties:*

In [11, 12] gluing between the extremal Reissner-Nordström and Bertotti-Robinson metrics [13–18] was considered as a classical model of an elementary particle that looks as a BH for an external observer but is regular inside. Let  $m$  be the ADM mass of such a BH,

$r_0$  being the radius of gluing. For  $r \geq r_0$  we have the extremal Reissner-Nordström metric (16), with  $B = (1 - \frac{m}{r})^2$ , and for  $r \leq r_0$  the metric has the form (30) with  $q = r_0$ . Then, as is shown in [11], the only nonvanishing component of the boundary surface stresses is equal to  $S_0^0 = \frac{\sqrt{B(r_0)}}{4\pi r_0} = \frac{\varepsilon}{4\pi r_0^2}$ , where  $\varepsilon = r_0 - m$ . For small but nonzero  $\varepsilon$  we have the configuration typical of a QBH: a static metric with the radius of the inner region arbitrarily close to that of the horizon. In the limit  $\varepsilon \rightarrow 0$  the quantity  $S_0^0 \rightarrow 0$ . For an extremal Reissner-Nordström BH the electric charge and the mass obey  $Q^{\text{RN}} = m$ . On the other hand, for the Bertotti-Robinson metric (30), one has  $Q^{\text{BR}} = q$ , so that in our case  $Q^{\text{BR}} = r_0$ . As a result, the shell separating two regions carries the charge  $Q^{\text{RN}} - Q^{\text{BR}} = -\varepsilon$  which also vanishes in the limit  $\varepsilon \rightarrow 0$ . Thus, in the static coordinate frame, in the quasihorizon limit  $\varepsilon = 0$ , one obtains that the surface stresses and the surface charge (that appear due to the gluing process between the two different metrics) vanish [11], so that the configuration becomes everywhere regular. For an outer observer, the corresponding spacetime reveals itself as an extremal BH but it is free of singularities inside (in contrast to the Reissner-Nordström metric) since the inner Reissner-Nordström core is replaced by the Bertotti-Robinson metric. One obtains a self-sustained configuration having no singular sources, which is balanced by its own forces without support from an external agent. In this sense, it can be considered as a classical electromagnetic model of an elementary particle, realizing Wheeler's idea of charge without charge [20].

#### *Tidal stresses and matter stresses:*

Now, let us see what happens in a freely falling frame. A free-falling frame reveals some new nontrivial features of the composite spacetime under discussion. Consider again the quantity  $Z = \bar{F} - \bar{K}$ , with  $\bar{K} = R_{\hat{0}\hat{0}}^{\hat{\theta}\hat{\theta}}$  and  $\bar{F} = R_{\hat{\theta}\hat{r}}^{\hat{\theta}\hat{r}}$ , which we introduced in Sec. III A while discussing some properties of extremal charged dust. In the free-falling frame the quantity  $\tilde{Z}$  is given by  $\tilde{Z} = Z(2\frac{E^2}{B} - 1)$ , where  $E$  is the energy of the particle and  $\tilde{Z}$  is calculated in the free-falling frame [21, 22] (see also [23–25]). For the Bertotti-Robinson metric one has  $Z \equiv 0$ , so that one obtains  $\tilde{Z} = 0$ . Therefore, one may wonder whether or not the naked behavior typical of other examples of QBHs occurs in this case. As we will show now, an analogue of naked behavior does indeed occur. Since now  $Z = 0$  there is no naked behavior

in the components of the Riemann tensor inside the boundary surface (i.e. in the Bertotti-Robinson region) but there is naked behavior in the components of the Ricci tensor (i.e., in the components of stresses) on the boundary surface itself. Let us see this in more detail. If one defines  $Y \equiv S_1^1 - S_0^0$ , then a local Lorentz boost leads to the expression  $\tilde{Y} = (2\frac{E^2}{B} - 1)Y$  since  $Y$  transforms like  $Z$ , with the  $\theta - \theta$  components being insensitive to radial boosts. For the system under consideration, the only nonvanishing component in the static frame is  $S_0^0 = \frac{\varepsilon}{4\pi r_0^2}$  (see above). As a result, in a free-falling frame  $\tilde{Y} = \frac{1}{4\pi r_0^2} \left( \varepsilon - 2\frac{E^2 r_0^2}{\varepsilon} \right) \sim \varepsilon^{-1}$ , which clearly diverges in the QBH limit,  $\varepsilon \rightarrow 0$ . Thus, we have displayed a remarkable result: for  $\varepsilon \neq 0$  the boundary stresses are finite and nonzero, both in the static and free-falling frames. However, in the limit  $\varepsilon \rightarrow 0$ , they disappear in the static frame, but go unbounded in the free-falling one. In this sense, the situation in the electrovacuum case is totally similar to that discussed above for the extremal dust and non-Abelian gauge systems. The only difference is that now the relevant quantities are not curvature components but boundary stresses.

## 2. Gluing between the extremal Reissner-Nordström and Minkowski metrics [19]

An even simpler example can be invoked, where gluing between an inner flat metric and an external extremal Reissner-Nordström metric is performed. Such a construction was discussed in [19] as an example of a classical model of an elementary particle (see also [7] and [11]). Consider an external spacetime given by equation (16) for  $r \geq r_0$ , and an inner spacetime given by the Minkowski metric,

$$ds^2 = -dT^2 + dr^2 + r^2 d\Omega^2 \quad (32)$$

where  $0 < r \leq r_0$ . On the border, the condition of matching both parts of the spacetime leads to

$$t = \frac{Tr_0}{r_0 - m}, \quad (33)$$

so that the time part of the metric (32) can be written as  $-dT^2 = -\frac{(r_0 - m)^2}{r_0^2} dt^2$ . Then, if time  $t$  is used, the metric coefficient  $g_{00} \rightarrow 0$  in the limit  $r_0 \rightarrow m$ . This is the reason why this construction can be considered as an example of a QBH. We again obtain an infinite

redshift due to the mismatch in time rescaling in equation (33). Also, we cannot achieve the continuous matching if  $T$  is considered as a legitimate coordinate inside since the surface  $r = m$  is timelike in the metric (32) but lightlike in the metric (16). One may try to repair this by considering inside the same time  $t$  as outside. However, the term  $-\frac{(r_0-m)^2}{r_0^2} dt^2$  disappears in this limit and the spacetime becomes degenerate. If one calculates the surface stresses on the boundary, it turns out that  $8\pi S_0^0 = -\frac{2}{m} \neq 0$  (all other components vanish) [11]. Then, reasonings from the previous subsection IIIC1 apply and we obtain a naked behavior on the shell in the limit under discussion for a radially infalling observer.

#### IV. A FURTHER PROPERTY: QUASI-BLACK HOLES SHOULD BE EXTREMAL

In all examples considered above the horizon approached by the system is extremal. One may ask, whether or not QBHs with nonextremal horizons are possible. In [29], with the help of numerical calculations, it was shown that, for some particular charge density and energy density distributions, the boundary of a body with  $q < m$  (where  $q$  is the total charge and  $m$  is the ADM mass) cannot approach its own horizon, the system collapses before reaching it. This result corroborates the Buchdahl limits, first worked out to the Schwarzschild interior solution, as well as for perfect fluid matter. In [30] an interesting, although convoluted, analytical proof generalizing the Buchdahl limits for charged perfect fluid was given. Here we state an even more general theorem, without resorting to the equation of state of the matter or other system's details at all.

The statement we want to prove is “a static regular configuration cannot approach its own horizon arbitrarily closely if the horizon is nonextremal.” The proof goes as follows. By definition, a nonextremal horizon (NEH) implies that the surface gravity  $\kappa$  is nonzero. Since  $\kappa = \left(\frac{d\sqrt{B}}{dl}\right)_h$ , where the derivative is taken on the horizon  $h$ , this condition gives  $\left(\frac{d\sqrt{B}}{dl}\right)_h \neq 0$ . We call this the NEH condition. Let conditions (a)-(c) of Sec. II be fulfilled, so that we have a QBH. Consider separately two cases, namely, (1)  $\sqrt{B(r)}$  has a continuous derivative in relation to the proper length  $l$ , i.e.,  $\frac{d\sqrt{B(r)}}{dl}$  is continuous, and (2)  $\sqrt{B(r)}$  is merely continuous, so that a surface layer is allowed.

- (1) When  $\sqrt{B(r)}$  has a continuous derivative in relation to  $l$ , i.e.,  $\frac{d\sqrt{B(r)}}{dl}$  is continuous, then also  $\frac{d\sqrt{B(r)}}{dr}$  is continuous. Thus we are assuming that  $\sqrt{B(r)}$  is of class  $C^1$ . Now we will show that the NEH condition and condition (c) are mutually inconsistent. Recall that condition (c) states that in the limit  $\varepsilon \rightarrow 0$  the metric coefficient  $B \rightarrow 0$  for all  $r \leq r^*$ . Let us exploit the following simple lemma, which we will prove shortly. Assume  $\sqrt{B}$  is any function such that the condition (c) is satisfied, and further assume (d)  $\sqrt{B} > 0$ , for  $\varepsilon \neq 0$  and  $r \leq r^*$ . Then in the limit  $\varepsilon \rightarrow 0$  one cannot get  $\frac{d\sqrt{B(r)}}{dl} \neq 0$ , as one should for a nonextremal BH. Now we prove this lemma. Let us suppose, for a moment, that  $\frac{d\sqrt{B}}{dr} \rightarrow a_0 \neq 0$  at some  $r_1$  where  $0 \leq r_1 < r^*$ . Using a Taylor expansion, we can write  $\sqrt{B} = a_0(r - r_1) + \dots$  in the vicinity of  $r_1$  for sufficiently small  $\varepsilon$ , with  $a_0$  a constant. For  $a_0 > 0$  we have that  $\sqrt{B} < 0$  for  $r < r_1$  in contradiction with condition (d). As well, for  $a_0 < 0$  we have that  $\sqrt{B} < 0$  for  $r > r_1$  in contradiction with condition (d). So the only possibility is  $\frac{d\sqrt{B}}{dr} \rightarrow a_0 = 0$ . As, by assumption, the derivative  $\frac{d\sqrt{B}}{dr}$  is continuous, we can extend this line of reasoning to some vicinity  $(r^* - \delta, r^* + \delta)$  of the boundary point  $r^*$ , take advantage of the Taylor series again, by the same reasoning obtain that  $\left(\frac{d\sqrt{B}}{dr}\right)_{r=r^*} \rightarrow 0$ , and so,  $\left(\frac{d\sqrt{B}}{dl}\right)_h \rightarrow 0$  as well.
- (2) When  $\sqrt{B(r)}$  is merely continuous, one is relaxing the condition of the continuity of the first derivative and thus allowing the existence of a surface layer. We will see now that it does no good. In this case we would have a deltalike term in the stress-energy tensor  $\tilde{T}_\mu^\nu$  and a nonzero Lanczos tensor  $S_\mu^\nu = \int dl \tilde{T}_\mu^\nu$ , where the integral is to be performed across the boundary. There is only one independent spatial component of the tangential stresses, namely  $S_2^2$ . This is given by,  $8\pi S_2^2 = \frac{(r)_+ - (r)_-}{r^*} + \frac{(\sqrt{B})'_+ - (\sqrt{B})'_-}{\sqrt{B}}$ , where the + and - signs refer to the outer and inner regions, respectively, and a prime denotes a derivative with respect to the proper distance  $l$ . The first term is finite and is equal to zero, if we do not put a finite mass on the surface  $r = r^*$ . However, the second term diverges since the numerator is finite whereas the denominator tends to zero. Thus, the boundary stresses become infinite and the configuration becomes strongly singular.

Thus, we see that in case (1) the condition of nonextremality cannot be fulfilled, and in case

(2) the condition of regularity fails. The proof of our statement is completed, there are no nonextremal QBHs.

On the other hand, if the surface gravity  $\kappa = 0$ , i.e., the QBH is extremal, the above arguments do not work since  $\frac{d\sqrt{B}}{dl} \rightarrow 0$  from both sides of  $r^*$ . As a result, in case (1) there is no contradiction between conditions NEH and (c), and in case (2)  $S_2^2$  can be finite. So QBHs can only be extremal.

## V. REGULAR VERSUS SINGULAR BEHAVIOR AND UNATTAINABILITY OF THE QUASI-BLACK HOLE LIMIT

Upon careful inspection, one finds that in QBHs divergencies on the Kretschmann scalar do not occur. However, the finiteness of this quantity is not the only criterion for regular or singular classification of a spacetime. One example is the behavior of naked BHs. Indeed, in some special frames the Riemann tensor diverges near the horizon of these naked BHs and these divergences can be related to nonscalar polynomial curvature singularities discussed in [26].

In the present work we have encountered a rather unusual entanglement of regular and singular features in QBHs. From the viewpoint of an external observer who uses time measured by clocks at infinity, an inner region looks like a degenerate spacetime with the component of the metric  $g_{00} \rightarrow 0$  everywhere. Yet, this singular feature has nothing to do with the behavior of the Riemann tensor. Its components in an orthonormal static frame are finite there, and the Kretschmann scalar is also well behaved. The most obvious manifestation of this property is the example discussed in Section III C 2 where the inner spacetime is flat, nonetheless it exhibits singular features! If one tries to remove the degeneracy of the inner spacetime by rescaling the time coordinate, another difficulty arises: the spacetime ceases to be continuous since the surface is lightlike from the viewpoint of an outer observer but is timelike from the viewpoint of an inner one. To put it in another way: one can easily achieve the validity of the matching conditions on a timelike surface, but if this surface tends to a null surface, at least from one side, the procedure ceases to be well-defined and this gives rise to a number of unusual properties. Another singular feature consists in the impossibility

to penetrate from the inside to the outside and vice versa. In this sense, geodesics cannot be extended across the border between different regions, in spite of the fact that each of them, taken by itself, can be extended. For instance, the Minkowski spacetime in Section III C 2 is obviously extendable but this extension has nothing to do with the problem under discussion in which the outer spacetime should be the extremal Reissner-Nordström BH. The fact that observers in different regions disagree about the border's nature, whether it is timelike or null, can be considered as one of the manifestations of the mutual impenetrability. Actually, it shows that one deals with two separate spacetimes. It turns out that there is some kind of complementary relationship between the inner and outer regions and between their regular and singular properties. If an observer is situated inside, he will say that the geometry is perfectly regular there but becomes singular on the border and beyond, so that he is unable to penetrate to outside. The outer observer, on the contrary, will say that it is his region which is regular (excepting the border) and finds he cannot penetrate into the inner singular region. All this forces us to conclude that the spacetime of a QBH as a whole may be singular in spite of the fact that the Kretschmann scalar diverges nowhere.

This discussion helps to elucidate an important additional question, of whether or not the QBH limit (whose properties we have discussed in detail) is attainable in some real physical process. For comparison, in the Reissner-Nordström geometry, taking formally the limit  $q \rightarrow m$ , one can obtain the extremal Reissner-Nordström BH from the nonextremal one but, according to the third law of BH thermodynamics [27], this cannot be accomplished in any real process for a finite number of steps. Furthermore, if the cosmic censorship conjecture is valid, one cannot convert the BH state with  $q \leq m$  into a naked singularity by increasing the charge to  $q > m$ . What is said above about singular features in the QBHs properties leads to the conclusion that the corresponding limiting state is unattainable physically from any close regular configuration. More precisely, the state which is obtained by the mathematical procedure of taking the QBH limit can be approached as closely as one likes. However, if we assume that regular configurations cannot be turned into singular ones, the QBH limit cannot be attained by gradually changing an initial regular configurations to a singular one. A usual horizon hides singularities beyond it, but its analogue, the quasihorizon, in a sense, brings about certain singular features into the system. If these singular features

cannot arise by physical processes, as we have argued, this means that we are faced with a somewhat unusual counterpart of the cosmic censorship. On the face of what has been said, it seems that QBHs should extend the taxonomy, not only of relativistic objects, but also of singularity types in general relativity.

It is also worth remarking that in some cases the limiting configuration may turn out to be geodesically complete and regular like the manifold given by equation (27), obtained from the inner core region (i.e., region I), in Section III A. In this case, nothing prevents one from taking the limit  $c = 0$  in which, equation (27) arises from equation (20). In addition, the proper distance to the quasihorizon tends to infinity in this limit. Thus, it seems that the limit can be attainable in some regions and unattainable in others, which is one more unusual feature of QBHs.

Summing up, configurations that approach as close as one likes a QBH state can be easily achieved, and in this sense, QBHs may have real physical significance. But whether a QBH state can be attained in nature, through such a process, or perhaps emerge via some quantum process, is a thorny issue that certainly needs further investigation.

## VI. CONCLUSIONS

The present work unifies in the same QBH context seemingly different systems like those considered in [3–8] on Bonnor stars, in [9, 10] on magnetic monopoles, and in [11]–[20] on glued vacua. The properties of QBHs were worked out in some detail. It is then clear, that for an external static observer, a BH and a QBH look similar. Nevertheless, their inner nature is different. First, not only the outer original region is inaccessible for the inner observer, like in the BH case, but also vice versa, which has no analogue in the BH case. Second, while for BHs the separation of different regions is of pure causal nature, in the QBH it is dynamic rather than purely causal. The reasons for no penetration from one region to another are quite different, namely rescaling of time, and infinite tidal forces or infinite surface stresses, i.e., naked behavior. In addition, as far as the naked behavior is concerned, it is also worth noting for comparison that in all examples considered in [21–25] the curvature components in the free-falling frame are enhanced with respect to the static



value but remain finite, whereas for QBHs those components diverge. Thus, if a system is able to withstand gravity forces up to a state which is arbitrarily close to an extremal BH and not collapse, its inner properties, the QBH properties, are qualitatively distinct from those of a corresponding extremal BH. However, for a distant observer to distinguish between a QBH and an extremal BH might be virtually impossible.

As a last remark, we note that in the above considerations, we tacitly implied that the areal radius increases monotonically with the proper distance. Meanwhile, we can try to glue two copies of the spacetime in the spirit of cut and paste technique used in physics of wormholes [38] with the increasing and decreasing branches of the function  $r(l)$  and, afterward, take the QBH limit. For example, one can use the extended Bonnor star distributions described above. The corresponding limit possesses interesting properties that, however, needs a separate discussion. In [39] a special type of wormhole was considered. Interestingly enough, this wormhole can be considered as a system with properties somehow similar to those of a QBH, in the sense that it is connected with the threshold of the formation of a horizon, in this case nonextremal, from a wormhole configuration. Detailed comparison of the two approaches, based on near-extremal and nonextremal wormhole configurations, and properties of the corresponding spacetimes will be done elsewhere.

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## APPENDIX A: OBTAINING THE BERTOTTI-ROBINSON SPACETIME AS A LIMIT OF DIFFERENT METRICS

It is instructive to note that the Bertotti-Robinson spacetime [13–18] can appear as a result of two different limiting procedures.

(1) First, starting with the extremal Reissner-Nordström metric one obtains the Bertotti-Robinson metric by means of a well known limiting procedure [16]. Indeed, in the extremal Reissner-Nordström case one can make the transformation, from the usual Schwarzschild coordinate  $r$  to the proper radial coordinate  $l$ , given by,  $r = q + \lambda \exp\left(\frac{l}{q}\right)$ , and from  $t$  again to  $T$  given by,  $t = \frac{qT}{\lambda}$ , where  $\lambda$  is a parameter, and take the limit  $\lambda \rightarrow 0$ . Then the metric takes the form (30). In the course of this limiting transition the metric coefficient  $g_{00}^{\text{RN}}$  of the original extremal Reissner-Nordström metric tends to zero just due to taking the limit in the coordinate space: since  $\lambda \rightarrow 0$  we have  $r \rightarrow q$  and  $g_{00}^{\text{RN}}(r) \rightarrow g_{00}^{\text{RN}}(q) = 0$ . In the resulting Bertotti-Robinson manifold (30) the coefficient  $g_{00}^{\text{BR}} \neq 0$ , due to the factor  $\lambda^{-2}$  which compensates  $\lambda^2$  in  $g_{00}^{\text{RN}}$ . In doing so, the horizon of the original Reissner-Nordström metric ( $r = q$ ) maps into the horizon of the Bertotti-Robinson metric ( $l = -\infty$ ).

(2) Second, in the QBH case discussed in section III A 2, the reason why  $g_{00} \rightarrow 0$  comes from taking a special limit in the space of parameters: we have  $g_{00}^{\text{QBH}} = g_{00}^{\text{QBH}}(r, c)$  and  $g_{00}^{\text{QBH}}(r^*, c) \neq 0$ , for  $c \neq 0$ . But  $\lim_{c \rightarrow 0} g_{00}^{\text{QBH}}(r^*, c) = 0$  where  $r^*$  corresponds to the quasihorizon [8]. In doing so, the quasihorizon  $r = r^*$  corresponds to  $l = 0$ , i.e.,  $\eta = 1$  in (29). Then, it is seen from (30) that  $g_{00}^{\text{BR}} \neq 0$  at  $\eta = 1$  and, thus, this value of  $\eta$  does not correspond to the horizon of the Bertotti-Robinson metric. The horizon of the metric (30) lies at  $l = -\infty$  where  $g_{00}^{\text{BR}} \rightarrow 0$ . In other words, the quasihorizon of the original metric (20) does not map onto the horizon of the Bertotti-Robinson obtained from it through the limiting procedure. Instead, the transformation (29) in the limit  $c \rightarrow 0$  maps the origin  $r = 0$  of (20) onto the horizon of the metric (30) (which does not possess an origin at all) since in this limit  $r^* \sqrt{B} \sim c^{2/3}$  and  $\eta \sim c^{1/3} \rightarrow 0$  (see [8]).

Thus, we see that although in region II our metric takes the Bertotti-Robinson form, it cannot be considered as a trivial consequence of the known limiting procedure from the extremal Reissner-Nordström metric. Cases (1) and (2) are different and map the horizon,

and the quasihorizon, of the original manifold in different manners.

## APPENDIX B: PROOF THAT $\frac{d\sqrt{B}}{dr} \geq 0$ FOR QUASI-BLACK HOLES

One general property of QBHs is that the metric function  $\sqrt{B}$ , for the systems under discussion, obeys the condition

$$\frac{d\sqrt{B}}{dr} \geq 0. \quad (\text{B1})$$

Indeed, assuming that the Einstein equations are satisfied, one has

$$\frac{1}{\sqrt{B}} \frac{d\sqrt{B}}{dr} = \frac{m + 4\pi p_r r^3}{r(r - 2m)}, \quad (\text{B2})$$

where  $m(r)$  is the total gravitational mass enclosed inside the radius  $r$ , and  $p_r$  is the total radial pressure, arising from all the fields and matter that may be present. For regular matter configuration there are no horizon, so the denominator is positive. In addition, the numerator is positive for systems with  $m(r) + 4\pi p_r r^3 > 0$ . The known Majumdar-Papapetrou exact solutions show that (B2) holds for these systems [3–8]. For the self-gravitating monopole its validity is clearly seen from numeric calculations [9, 10]. For the composite vacuum systems studied here [11, 12], composed of Reissner-Nordström and Bertotti-Robinson geometries, the situation is more tricky, as the coordinate  $r$  becomes degenerate, but the positivity of (B2) is guaranteed upon a suitable redefinition of distance.

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